IMM-NYU 261
October 1959

NEW YORK UNIVERSITY
INSTITUTE OF
MATHEMATICAL SCIENCES

SPACES OF RIEMANN SURFACES

LIPMAN BERS

PREPARED UNDER
CONTRACT NO. DA-30-069-ORD-2153
WITH THE
OFFICE OF ORDNANCE RESEARCH
U.S. ARMY

IMM-NYU 261
October 1959

New York University
Institute of Mathematical Sciences

SPACES OF RIEMANN SURFACES*

Lipman Bers

* Address delivered at the International Congress of Mathematicians at Edinburgh in August 1958 by invitation of the Organizing Committee. To appear in the Proceedings of the Congress.

This report represents results obtained at the Institute of Mathematical Sciences, New York University, under the sponsorship of the Office of Ordnance Research, U.S. Army, Contract No. DA-30-069-ORD-2153.

SPACES OF RIEMANN SURFACES

By LIPMAN BERS

This address is a progress report on recent work, partly not yet published, on the classical problem of moduli. Much of this work consists in clarifying and verifying assertions of Teichmüller[23-28] whose bold ideas, though sometimes stated awkwardly and without complete proofs, influenced all recent investigators, as well as the work of Kodaira and Spencer on the higher dimensional case. Following Teichmüller we consider not the space of closed Riemann surfaces of a given genus g but rather an appropriable covering space and certain related spaces. For the sake of brevity the simple and somewhat exceptional cases $g = 0$ and $g = 1$ will be omitted.

Our main technical tools are uniformization theory and the theory of partial differential equations. The problem of moduli has also an algebraico-geometrical aspect, but the topological and analytical methods used here are, of course, restricted to the classical case. On the other hand, they are, in principle, applicable also to open surfaces.

1. Quasiconformal mappings

Let $w = w(x) = u(x,y) + iv(x,y)$ be a homeomorphism of a domain $\mathcal{D}$ in the z-plane onto a domain in the w-plane, and let k be a number such that $0 \leq k < 1$; we set $K = (1 + k)/(1 - k)$. There exist three distinct ways of defining k quasiconformality of the mapping w.

<u>Definition</u> A (Morrey[17], Caccioppoli[8], Bers and Nirenberg[6]). The derivatives w_x, w_y exist as generalized L_2

derivatives and almost everywhere

$$|w_x + iw_y| \leq k|w_x - iw_y|. \tag{1}$$

For a C_1 mapping w with positive Jacobian this is the original definition used by Grötzsch[10-12], Ahlfors[1] and Teichmüller[23]. We recall that w_x is called a generalized L_2 derivative of w if w and w_x are measurable and locally square integrable in $\mathcal{D}$ and $\int w_x \phi \,dxdy = -\int w\phi_x \,dxdy$ for every C_{co} function ϕ with compact support in $\mathcal{D}$.

<u>Definition</u> B (Ahlfors[2], Pfluger[19], Mori[16]). For every topological rectangle $\mathcal{R} \subset \mathcal{D}$

$$\operatorname{mod} w(\mathcal{R}) \leq K \operatorname{mod} \mathcal{R}. \tag{2}$$

We recall that a topological rectangle $\mathcal{R}$ is a conformal image of a closed rectangle $0 \leq \xi \leq 1$, $0 \leq \eta \leq m$, and $\operatorname{mod} \mathcal{R} = m$.

<u>Definition</u> C (Lavrent'ev[15], Pesin[18], Jenkins[13]). At almost all points z of $\mathcal{D}$

$$\limsup_{r \to 0} \left\{ \max_{|z-\zeta|=r} |w(z) - w(\zeta)| \Big/ \min_{|z-\zeta|=r} |w(z) - w(\zeta)| \right\} \leq K.$$

That quasiconformality is a natural concept is shown by the

Equivalence Theorem. <u>Each of the three definitions A, B, C implies the other two.</u>

The implication A -> B was proved by Grötzsch for C_1 mappings; his proof extends to the general case in view of the results of Morrey. Mori's work contains implicitly the statements B -> A, B -> C, cf. Bers[4]. Pesin and Jenkins showed that C -> A. (Cf. also Volkoviskii[29], Yujobo[31].)

A k quasiconformal mapping remains so if followed or preceded by a conformal mapping. Hence we may define a homeomorphism f of a Riemann surface S onto another such surface S' to be quasiconformal if it is so in a neighborhood of every point on S, in terms of local parameters.

2. Beltrami equations

It follows from Definition A that every k quasiconformal mapping of a plane domain satisfies a Beltrami equation

$$w_x + iw_y = \mu(z)(w_x - iw_y) \quad (|\mu| \leq k < 1), \qquad (3)$$

where μ is a complex-valued measurable function. Conversely, every homeomorphic solution (with generalized L_2 derivatives) of (3) is k quasiconformal. We recall the geometric meaning of (3): the mapping $z \to w(z)$ is conformal with respect to the metric $ds = |dz + \mu d\bar{z}|$.

Let M denote the set of all bounded measurable functions $\mu(z), |z| < 1$, with $\|\mu\|$ = true max $|\mu(z)| < 1$. We topologize M by requiring $\mu_j \to \mu$ to mean that $\|\mu_j\| \leq k_0 < 1$ and $\mu_j(z) \to \mu(z)$ a.e. For $\mu \varepsilon M$ let w^μ denote a solution of (3) which maps $|z| \leq 1$ topologically onto itself leaving the points $1, i, -1$ fixed.

Proposition I (Morrey, cf. Bers and Nirenberg[6], Boyarskiĭ[7]). For $\mu \varepsilon M$, w^μ <u>exists and is unique and every other solution of</u> (3) <u>is an analytic function of</u> w^μ. Also w^μ and $(w^\mu)^{-1}$ <u>satisfy uniform Hölder conditions depending on</u> $\|\mu\|$.

Proposition II. *Let* $\mu \in M$ *depend on several real parameters* $t_1, \ldots, t_r$ *and be a function of class* C_ν $(\nu = 0,1,2, \ldots,\infty)$ *of these parameters. For every* z, $|z| \leq 1$, $w^\mu(z)$ *is of class* C_ν *as a function of* $T_1, \ldots, t_r$.

Proposition III. *Let* $\mu \in M$ *depend holomorphically on several complex parameters* $s_1, \ldots, s_r$. *For a sufficiently small* $\varepsilon > 0$ *there exists a homeomorphic solution* $w(z)$ *of* (3) *defined for* $|z| < \varepsilon$, *such that* $w(z)$ *is a holomorphic function of* $s_1, \ldots, s_r$.

The proofs of II and III will appear elsewhere.

3. Teichmüller spaces

In what follows conformally equivalent Riemann surfaces are considered identical. Two Riemann surfaces S and S_0 will be called *similar* if there exists a quasiconformal homeomorphism f of S onto S_0. In this case the homotopy class F_{S,S_0} of f is called allowable and the pair (S,F_{S,S_0}) is called a marked Riemann surface. The totality of these forms the *Teichmüller space* $T(S_0)$. Every allowable class F_{S_1,S_0} *defines* in an obvious way a one-to-one mapping (allowable mapping) of $T(S_1)$ onto $T(S_0)$. We are interested only in properties invariant under allowable mappings; hence we may identify $T(S_1)$ with $T(S_0)$ and call it the Teichmüller space T determined by a class of similar Riemann surfaces.

A differential of type (p,q) on S_0 is, locally, of the form $\lambda(z)dz^p d\bar{z}^q$, where z is a local parameter and $\lambda(z)$ a measurable

function. Let $m = \mu d\bar{z}|dz$ be a differential of type $(-1, 1)$ (Beltrami differential). Then $|\mu|$ is a scalar; if $\|m\| = \text{true max } |\mu| < 1$, m is called a proper Beltrami differential. It defines on S_o a Riemannian metric

$$ds = |dz + \mu d\bar{z}|$$

and it follows from I that this metric defines on S_o a new conformal structure. S_o with this conformal structure and with the allowable class containing the identity is a marked Riemann surface which we denote by S_o^m. Every element of $T(S_o)$ is of the form S_o^m, but $S_o^{m_1} = S_o^{m_2}$ (equality in the sense of marked Riemann surfaces) does not imply that $m_1 = m_2$.

The Teichmüller distance between two elements of $T(S_o)$, say S_1 and S_2, is defined as $\inf \|m\|$ for all m such that $S_1 = S_2^m$. It defines a topology on $T(S_o)$.

Now let S_o be neither the sphere, nor the plane, nor the cylinder, nor a closed surface of genus 1. Then we have the representation $S_o = U|G_o$, where U is the unit disc and G_o a Fuchsian group (by which we mean here a discrete fixed-point-free group of non-Euclidean motions). G_o is determined by S_o uniquely, except that it may be replaced by AG_oA^{-1}, where A is a non-Euclidean motion.

Let M_{G_o} denote the set of those $\mu \in M$ which satisfy the functional equation

$$\mu(A(z))\, \overline{A'(z)}/A'(z) = \mu(z) \quad (\text{for } A \in G_o). \qquad (4)$$

Every Beltrami differential on S_o can be written as $m = \mu(z)d\bar{z}/dz$, $|z| < 1$, $\mu \in M_{G_o}$. For $\mu \in M_{G_o}$ one verifies (using I) that

$$w^{\mu}(A(z)) = A^{\mu}(w^{\mu}(z)) \quad \text{for} \quad A \in G_o, \tag{5}$$

where A^{μ} is a non-Euclidean motion. The mapping $A \to A^{\mu}$ is an isomorphism of G_o onto a Fuchsian group $G_o^m = w^{\mu}G_o(w^{\mu})^{-1}$. We have that $S_o^m = U/G_o^m$. Thus the study of Teichmüller spaces can be made dependent on the theory of Beltrami equations.

4. <u>The spaces</u> T_g, $T_{g,n}$, $T_g^{(n)}$

Let S_o be a closed Riemann surface of genus g. Every closed surface S of genus g is similar to S_o and every sense-preserving homeomorphism of S onto S_o <u>belongs to an allowable class</u>. The proof of this is not difficult in view of our definition of quasiconformality. The Teichmüller space of S_o will be denoted by T_g.

Analogous statements are true if S_o and S are each obtained by removing n distinct points from a closed Riemann surface of genus g. The corresponding Teichmüller space will be denoted by $T_{g,n}$.

We shall also consider (for $g > 1$) the space $T_g^{(n)}$ the elements of which are marked closed Riemann surfaces of genus g on each of which one has distinguished an ordered n-tuple of (not necessarily distinct) points.

There is a natural mapping ϖ of $T_g^{(n)}$ onto T_g and the inverse image of a point of T_g under ϖ is in a one-to-one correspondence with the n-fold product of a Riemann surface by itself. This remark yields a natural way of introducing a topological

or differentiable structure in $T_g^{(n)}$ once we have such a structure in T_g. We denote by $\hat{T}_g^{(n)}$ the set of points of $T_g^{(n)}$ corresponding to the choice of n distinct points on a surface. There is a natural mapping π of $T_{g,n}$ onto $\hat{T}_g^{(n)}$ which permits one to define a topological or differentiable structure in $T_{g,n}$ using the corresponding structure of $T_g^{(n)}$. (π depends upon an arbitrary ordering of the 'removed' points on one element of $T_{g,n}$.)

5. Embedding of T_g into E_{6g-6}

In what follows, we consider a fixed g and assume, for the sake of brevity, that $g > 1$. We set $\tau = 3g - 3$.

Let $S_o = U/G_o$ be a closed surface of genus g. It is known That G_o consists of the identity and of non-Euclidean translations. Thus every element $A \neq 1$ of G_o has exactly two fixed points on the unit circle and one can show that two distinct elements have four distinct fixed points. Also, one can choose 2g generators A_j^o, B_j^o, $j = 1,2, \ldots, g$ of G_o satisfying the relation $\prod A_j^o B_j^o (A_j^o)^{-1} (B_j^o)^{-1} = 1$ (standard set of generators). We call a standard set normalized if the repelling and attracting fixed points of B_g are 1 and (-1), respectively, and one of the fixed points of A_g is i. We assume that a definite normalized standard generating set of G_o has been chosen once and for all. (This can always be done, replacing if need be G_o by AG_oA^{-1}.)

Now set, for some $\mu \in M_{G_o}$,

$$A_j = w^\mu A_j^o (w^\mu)^{-1}, \quad B_j = w^\mu B_j^o (w^\mu)^{-1} . \qquad (6)$$

Then $\{A_j, B_j\}$ is a normalized standard set of generators for G^m and hence determines S^m. Moreover, $\{A_j, B_j\}$ depends only on S^m and not on m (for homotopic mappings induce homomorphisms of fundamental groups which differ only by inner automorphisms, and there are canonical mappings of G onto the fundamental groups of U/G). Finally, for a standard normalized set, A_g and B_g can be computed from $A_1, B_1, \ldots, B_{g-1}$ by using the relation

$$\prod_{j=1}^{g} A_j B_j A_j^{-1} B_j^{-1} = 1 \quad . \tag{7}$$

Each A_j and B_j, $j = 1, \ldots, g-1$ can be represented by 3 real numbers, Thus we can represent every element of $T(S_o) = T_g$ by a point in the Euclidean space E_{6g-6}. From now on we identify an element of T_g with its representative point. Now T_g appears as a subset of E_{6g-6}, and hence is topologized and even metrized (cf. Siegel[22], Bers[5]).

6. Differentiable structure of T_g

Lemma 1. Let S_o be a marked closed Riemann surface of genus g, $m = (m_1, \ldots, m_\sigma)$ a σ- tuple of Beltrami differentials on S_o, $\xi = (\xi_1, \ldots, \xi_\sigma)$ a point of E_σ of small modulus $|\xi|$. The mapping $\xi \to S^{\xi\cdot m} = S_o^{\xi_1 m_1 + \ldots + \xi_\sigma m_\sigma}$ of a neighborhood of 0 in E_σ into E_{6g-6} is C_∞.

This follows at once from II.

In a forthcoming paper Ahlfors and Bers prove that the mapping considered is even real analytic.

A regular quadratic differential Ω on S_o is locally of the form $\omega(z)dz^2$, $\omega(z)$ holomorphic. These Ω's form a complex

vector space Q_{S_o} of dimension $3g - 3 = \tau$ (Riemann-Roch). We note that for any Beltrami differential $m = \mu d\bar{z}/dz$ the scalar product

$$(\Omega, m) = \iint_{S_o} \omega(z)\mu(z)dxdy$$

is well defined.

A Beltrami differential m on S_o will be called <u>locally trivial</u> if for real $\varepsilon \to 0$

$$|S_o - S_o^{\varepsilon m}| = o(\varepsilon). \tag{8}$$

To appreciate this requirement, note that by II we always have

$$|S_o - S_o^{\varepsilon m}| = O(\varepsilon).$$

The following result goes back to Teichmüller.

<u>Main Lemma.</u> <u>The Beltrami differential</u> m <u>on</u> S_o <u>(closed Riemann surface of genus</u> g) <u>is locally trivial if and only if</u> $(\Omega, m) = 0$ for all $\Omega \in Q_{S_o}$.

<u>Necessity proof</u> (à la Ahlfors). Using II we compute that, for every A in G_o, $\dot{A} = (\partial A^{\varepsilon\mu}/\partial \varepsilon)_{\varepsilon=0}$ equals $h(A(z)) - A'(z)h(z)$, where

$$h(z) = (\partial w^{\varepsilon\mu}(z)/\partial \varepsilon)_{\varepsilon=0} \quad \text{and} \quad h_{\bar{z}} = \mu.$$

Equation (8) implies that $\dot{A} = 0$, i.e. that $h(z)/dz$ is a differential of type $(-1,0)$ on S_o and

$$(\Omega, m) = \iint_{S_o} \omega\mu dxdy = \iint_{S_o} \omega h_{\bar{z}} dxdy = 0$$

since $\omega_{\bar{z}} \equiv 0$.

Sufficiency proof (à la Weil). Let $\Omega_1, \ldots, \Omega_\tau$ be a complex basis of Q_{S_o}, L some C_{∞} density on S_o (i.e. a differential of type (1,1), $L = i\lambda dz d\bar{z}$ with $\lambda \geq 0$) and set

$$m_j = \bar{\Omega}_j/L, \quad j = 1, \ldots, \tau, \quad m_j = -i\Omega_{j-\tau}/L, \quad j = \tau+1, \ldots, 2\tau.$$

Assume that $(\Omega_j, m_o) = 0$ for all j and consider the mapping of $E_{2\tau+1}$ into $E_{2\tau}$:

$$(\xi_o, \xi_1, \ldots, \xi_{2\tau}) \to S^{\xi_o m_o + \xi_1 m_1 + \cdots + \xi_{2\tau} m_{2\tau}}.$$

This mapping has rank $\leq 2\tau$ at the origin. Hence there is a $(2\tau + 1)$-tuple $(\xi_o, \ldots, \xi_{2\tau}) \neq (0, \ldots, 0)$ for which $\xi_o m_o + \cdots + \xi_{2\tau} m_{2\tau}$ is locally trivial. But then $\xi_1 = \cdots = \xi_{2\tau} = 0$, by the result proved above, so that $\xi_o \neq 0$ and m_o is locally trivial.

A real (complex) Beltrami basis on S_o is a basis of the real (complex) factor-space of all Beltrami differentials modulo the locally trivial ones. The preceding argument contains the proof of

Lemma 2. *Let $m = (m_1, \ldots, m_{2\tau})$ be a real Beltrami basis on S_o. The mapping $\xi = (\xi_1, \ldots, \xi_{2\tau}) \to S_o^{\xi \cdot m}$ has rank 2τ at the origin.*

Corollary. *T_g is an open subset of E_{6g-6}.*

Indeed, S_o is not distinguished from any other element of T_g.

We have now defined a C_{∞} structure in T_g and hence also in $T_g^{(n)}$ and in $T_{g,n}$.

7. Extremal quasiconformal mappings

A Teichmüller differential on a closed Riemann surface S_o of genus g is either 0 or a Beltrami differential of the

form $\kappa\bar{\Omega}/|\Omega|$ where Ω is a regular quadratic differential and κ a number, $0 \leq \kappa < 1$. These differentials have an extremal property proved correctly in Teichmüller's 1940 paper (cf. also Ahlfors[2], Bers[5]).

Theorem A. If m_0 is a Teichmüller differential and m any other Beltrami differential on S_0 (a closed surface of genus $g > 1$), and if $S_0^m = S_0^{m_0}$, then either $m = m_0$ or $\| m\| > \| m_0\|$.

We can now state the Teichmüller theorem for closed surfaces.

Theorem B. Let S_0 be a marked closed Riemann surface of genus $g > 1$. Every element S_1 of $T_g = T(S_0)$ admits the unique representation $S_1 = S_0^m$ where m is a Teichmüller differential.

The theorem means that every homeomorphism of S_0 onto S_1 can be deformed into a unique extremal one, which deviates least from conformality and which can be represented, locally, except near finitely many points, as a conformal mapping followed by a uniform stretching and then by another conformal mapping.

Now let S_0 be a surface obtained from a closed Riemann surface Σ_0 of genus g by removing n distinct points $p_1, \ldots, p_n$. A Teichmüller differential on S_0 is either 0 or a Beltrami differential of the form $\kappa\bar{\Omega}/|\Omega|$, where $0 \leq \kappa < 1$ and Ω is a quadratic differential which is holomorphic on Σ_0 except perhaps at the points p_j at which it may have simple poles.

Theorem C. Let S_0 be an element of $T_{g,n}$. Every other element S_1 of $T_{g,n}$ admits the unique representation $S_1 = S_0^m$, where m is a Teichmüller differential on S_0.

This Teichmüller theorem can be derived from B (see Ahlfors[2] for details). Theorem A was proved by Teichmüller in [25],

another proof is due to Ahlfors[2]. We sketch below the proof in Bers[5]. It differs from Teichmüller's own only technically.

Let $\Lambda = (\Lambda_1, \ldots, \Lambda_{6g-6})$ be a real basis of Q_{S_0}. For $x \in E_{6g-6}$, $|x| < 1$ set $\gamma(x) = S_0^{|x|x\cdot\bar\Lambda/|\cdot\Lambda|}$ if $x \neq 0$, $\gamma(0) = S_0$. The mapping $x \to \gamma(x)$ of $|x| < 1$ into $T_g \subset E_{6g-6}$ is continuous (by II) and one-to-one (by A), hence open and topological (by the theorem on the invariance of domain). We prove that it is onto. For $S_1 \in T_g$ there is a $\mu \in M_{G_0}$ with $S_1 = S_0^m$. For $0 \leq t \leq 1$ we have that $t\mu \in M_{G_0}$. Let Θ be the set of those t for which $S_0^{tm} = \gamma(x)$ for some x. Then Θ is open, by the previous result and contains $t = 0$. We must show that Θ is closed (so that $t = 1$ belongs to it). But by A we have that if $S_0^{tm} = \gamma(x)$, then

$$|x| \leq \| tm\| = t\| m\| \leq \| m\| < 1.$$

The closure of Θ follows by the local compactness of E_{6g-6} and the continuity of γ.

The argument just given also establishes the following results:

<u>Theorem</u> D. T_g <u>is a</u> $(6g - 6)$ <u>cell</u>; $T_{g,n}$ <u>is a</u> $(6g - 6 + 2n)$ <u>cell</u>.

<u>Theorem</u> E. <u>The Teichmüller metric in</u> T_g <u>and in</u> $T_{g,n}$ <u>yield the same topology as the embedding of</u> T_g <u>into</u> E_{6g-6}.

The statement that T_g is a $(6g-6)$ cell is already contained in the work of Fricke[9]. Fricke's proof is quite different and very difficult to follow

8. <u>Complex-analytic structure of T_g</u>

The existence of a 'natural' complex analytic structure in T_g has been asserted by Teichmüller[23]; the first proof was given by Ahlfors[3] after Rauch[21] showed how to introduce complex-analytic co-ordinates in the neighborhood of any point

of T_g which is not a hyperelliptic surface. Other proofs are due to Kodaira-Spencer[17] and to Weil[30]. The proof sketched below gives explicitly a set of co-ordinates near every point of T_g.

Let S_o be an element of T_g, that is a marked closed Riemann surface of genus g, and $m = (m_1, \ldots, m_{3g-3})$ a complex Beltrami basis. By Lemmas 1 and 2 the mapping $a = (a_1, \ldots, a_{3g-3}) \to S_o^{a\cdot m}$ is a C_∞ homeomorphism of a neighborhood of the origin of the complex number space C_{3g-3} onto a neighborhood of S_o. We call the a_j the co-ordinates associated with m.

Theorem F. The co-ordinates associated with complex Beltrami bases are complex-analytic co-ordinates in T_g.

It will suffice to prove two statements.

(i) If m and n are two complex Beltrami bases on S_o and the relation $a = a(b)$ is defined by the equation $S_o^{a\cdot m} = S_o^{b\cdot n}$, then $\partial a_i/\partial \bar{b}_j = 0$ at $a = b = 0$.

(ii) If m is a complex Beltrami basis on S_o, and $S_1 = S_o^{c\cdot m}$ with $|c|$ sufficiently small, then there exists a complex Beltrami basis n on S_1 such that if $a(b)$ is defined by the equation $S_o^{(c+a)\cdot m} = S_1^{b\cdot n}$, then

$$\frac{\partial a_i}{\partial b_j} = \delta_{ij} \quad \text{and} \quad \frac{\partial a_i}{\partial \bar{b}_j} = 0 \quad \text{at } b = 0.$$

Proof of (i). From our definitions and Lemma 1 we conclude that for any two Beltrami differentials s and t on S_o

$$|S_0^{s+t} - S_0^s| = O(\|t\|) \quad \text{for} \quad s \text{ fixed}, \tag{9}$$

$$S_0^{\varepsilon s+\varepsilon t} - S_0^{\varepsilon s}| = o(\varepsilon) \quad \text{if} \quad t \text{ is locally trivial}. \tag{10}$$

Now we have $n = Qm + r$, where Q is a constant matrix and

$$r = (r_1, \ldots, r_{3g-3}),$$

r_j being locally trivial. It is plainly sufficient to consider the case $Q = I$, i.e. $n = m + r$. Using Lemma 1 and (10), we have that for small $|a|$ and $|b|$:

$$|a - b| = O(|S_0^{a.m} - S_0^{b.m}|) = O(|S_0^{b.m-b\cdot r} - S_0^{b\cdot m}|) = o(|b\cdot r|) = o(|b|),$$

so that $\partial a_i / \partial \bar{b}_j = 0$.

Proof of (ii). If s and t are Beltrami differentials on S_0,

$$\lambda_s(t) = \frac{1 + \bar{s}}{1 + s} \frac{t}{1 - |s|^2} \tag{11}$$

is a Beltrami differential on S_0^s and a direct computation based on (9) shows that

$$|S_0^{s+t} - (S_0^s)^{\lambda_s(t)}| = O(\|t\|^2) \quad \text{for fixed } s. \tag{12}$$

Now set $s = c.m$ and $n = \lambda_s(m)$. Then

$$|a| = O(|S_0^{s+a.m} - S_0^s|) = O(|(S_0^s)^{a\cdot n} - S_0^s|) + O(|a|^2).$$

This shows that n is a complex Beltrami basis on $S_0^s = S_1$. Next, if $S_0^{s+a\cdot m} = S_1^{b\cdot n}$, we have, for small a and b:

$$|a - b| = O(|S_1^{a\cdot n} - S_1^{b\cdot n}|) = O(|S_1^{a\cdot n} - OS_0^{s+a\cdot m}|)$$

$$= O(|(S_0^s)^{\lambda_s(a\cdot m)} - S_0^{s+a\cdot m}|) = O(\|a\cdot m\|^2) = O(|a|^2),$$

whence $\partial a_i/\partial b_j = \delta_{ij}$ and $\partial a_i/\partial \bar{b}_j = 0$.

The space T_g also has a 'natural' Hermitian metric $d\sigma$ defined (by Weil) as follows. For $S_0 \in T_g$, let L denote the Poincarè density on S_0 (if $S_0 = U/G_0$, then $L = dxdy/y^2$). Let $\{\Omega_j\}$ be a complex basis of Ω_{S_0} such that $(\Omega_j, \bar{\Omega}_k/L) = \delta_{jk}$, and set $m_j = \bar{\Omega}_j/L$, $m = (m_1, \ldots, m_\tau)$. Let a_j be the co-ordinates associated with the complex Beltrami basis m on S_0; then $d\sigma^2 = \sum |da_j|^2$ at the point $S_0 \in T_g$. Weil proved, by a computation, that the metric $d\sigma^2$ is Kählerian.

9. Complex analytic structure of $T_g^{(n)}$ and $T_{g,n}$

The results of this and the following section confirm and extend some of Teichmüller's assertion in [28]. They also show that the complex-analytic structure defined above is natural and coincides with that of Rauch-Ahlfors.

Let S_0 be a marked closed Riemann surface of genus $g > 1, p_1, \ldots, p_n$ points on S_0, not necessarily distinct, $\zeta_1, \ldots, \zeta_n$ local uniformizers on S_0 with $\zeta_j = 0$ at p_j, $m = (m_1, \ldots, m_\tau)$ a complex Beltrami basis on S_0, $a = (a_1, \ldots, a_\tau)$ a complex vector of small modulus. By a permanent uniformizer near p_j we mean a continuous function $z_j = w_j(a_1, \ldots, a_\tau, \zeta_j)$ which vanishes for $\zeta_j = 0$, is holomorphic in a for fixed ζ_j,

and is, for fixed a, a homeomorphic solution of the Beltrami equation in ζ_j with $\mu = \sum a_i \mu_i$ (where $m_i = \mu_i(\zeta_j) d\bar{\zeta}_j / d\zeta_j$). Clearly, $(a_1, \ldots, a_\tau, z_1, \ldots, z_n)$ are a set of complex co-ordinates, called distinguished co-ordinates of a neighborhood of $(S_0, p_1, \ldots, p_n)$ in $T_g^{(n)}$. The existence of permanent uniformizers follows from Proposition III.

Theorem G. The distinguished co-ordinates give $T_g^{(n)}$ a complex-analytic structure.

Assume now that the points $p_1, \ldots, p_n$ are distinct. By a complex Beltrami basis $m = (m_1, \ldots, m_{\tau+n})$ on $(S_0, p_1, \ldots, p_n)$ (also called an extended Beltrami basis) we mean a basis of the (complex) factor-space of all Beltrami differentials n on S_0 modulo those n for which $(\Omega, n) = 0$ whenever the quadratic differential Ω is regular on S_0 except perhaps for simple poles at the p_j. For $a = (a_1, \ldots, a_{\tau+n})$ small, $(S_0^{a.m}, p_1, \ldots, p_n)$ is a point of $T_{g,n}$. We call the a_j co-ordinates associated with m.

Theorem H. The co-ordinates associated with extended Beltrami bases are complex-analytic co-ordinates in $T_{g,n}$.

Theorem I. The natural mappings $\bar{\omega}$ (of $T_g^{(n)}$ onto T_g) and π (of $T_{g,n}$ into $T_g^{(n)}$ are holomorphic. $\bar{\omega}$ makes $T_g^{(n)}$ into a complex fibre-space (the fibres being n-fold products of a marked closed Riemann surface by itself). π makes $T_{g,n}$ into the universal covering space of $\hat{T}_g^{(n)}$.

The rather simple proofs of Theorems G, H, I are omitted due to lack of space.

10. Meromorphic functions on T_g, $T_{g,1}$ and $T_g^{(1)}$

Let us choose a canonical dissection on S_o. This gives us on every marked Riemann surface S similar to S_o a set of generators $A_1, B_1, \ldots, A_g, B_g$ for the fundamental group $\pi(S)$ (which are determined but for an inner automorphism and satisfy (7)) and hence also a one-dimensional homology basis. If ϕ is an Abelian differential on S, let (A_j, ϕ), (B_j, ϕ) denote the A_j and B_j periods of ϕ_j respectively. Also, let ω_j denote the Abelian differential of the first kind on S with $(A_i, \omega_j) = \delta_{ij}$, and set $p_{ij} = (B_i, \omega_j)$. Then the p_{ij} are functions on T_g.

We note that S may be considered as a complex analytic submanifold of $T_g^{(1)}$. The ratios $f_{ij} = \omega_i/\omega_j$ are functions on $T_{g,1}$ and on $T_g^{(1)}$. Finally, let ω_{ij} $(i \neq j)$ denote the Abelian differential of the third kind on S which has simple poles with residues 1 at the zeros of ω_i and with residues (-1) at the zeros of ω_j. The ratios $f_{ijk} = \omega_{ij}/\omega_k$ are functions on $T_{g,1}$ and on $T_g^{(1)}$.

Theorem J. The p_{ij} are meromorphic on T_g. The f_{ij} and f_{ijk} are meromorphic on $T_{g,1}$ and on $T_g^{(1)}$.

We omit the proof which is based on Proposition II.

Let Φ denote the function field generated by the f_{ij}, f_{ijk}, and $\Phi_o \subset \Phi$ the field generated by the f_{ij}. It is known that every meromorphic function f on S belongs to Φ (and even to Φ_o, if S is not hyperelliptic). Hence f is a restriction of a meromorphic function defined on the whole space $T_g^{(1)}$.

11. Applications

The following two results can be proved in a few lines using the fact that T_g is connected.

(a) Let A_j, B_j, $j = 1,\ldots,g$, be non-Euclidean motions generating, with the single relation (7), a fixed-point-free Fuchsian group with compact fundamental region. Represent A_j, B_j by (2x2) matrices α_j, β_j. Then $\prod \alpha_j \beta_j \alpha_j^{-1} \beta_j^{-1} = +I$ (identity matrix) and not $(-I)$. This answers a question of Siegel[22].

(b) Every canonical dissection of a closed Riemann surface $S = U/G$ can be deformed into a dissection which maps into a convex non-Euclidean polygon in U. This was stated, with a different and complicated proof, by Fricke[9].

12. Open questions

Here are some open questions.

(1) The space of (unmarked) Riemann surfaces is the factor-space T_g/Γ_g, Γ_g being the so-called mapping class group. Give a precise description of this space.

(2) Does there exist a complex-analytic co-ordinate patch covering T_g? (In other words is T_g a subset of C_{3g-3}?)

(3) How can the theory sketched above be extended to open surfaces (other than those obtained from a closed surface by removing points or disks)? Our definition of Teichmüller spaces tries to anticipate such an extension.

(4) In particular, if S_0 and S are two similar marked open Riemann surfaces, what is the nature of the extremal quasi-conformal mapping of S onto S_0, and is this mapping unique?

R E F E R E N C E S

[1] Ahlfors, L. V. Zur Theorie der Überlagerungsflächen. Acta Math. 65, 157-194 (1935).

[2] Ahlfors, L. V. On quasi-conformal mappings. J. Anal. Math. 3, 1-58 (1953-4).

[3] Ahlfors, L. V. The complex analytic structure of the space of closed Riemann surfaces. (To appear.)

[4] Bers, L. On a theorem of Mori and the definition of quasiconformality. Trans. Amer. Math. Soc. 84, 78-84 (1957).

[5] Bers, L. Quasiconformal mappings and Teichmüller's theorem. (To appear.)

[6] Bers, L. and Nirenberg, L. On a representation theorem for linear elliptic systems with discontinuous coefficients and its applications. Atti del Convegno Internazionale sulle Quazioni alle derivate parziali, Trieste, 1954, pp. 111-140 (1955).

[7] Boyarskiǐ, B. V. Homeomorphic solutions of Beltrami systems. Dokl. Akad. Nauk, SSSR, 102, 661-664 (1955). (Russian.)

[8] Caccioppoli. Fondamenti per una teoria generale delle funzioni pseudo-analitiche di una variable complessa. Rend. Acad. Naz. Lincei, 13, 197-204, 321-329 (1952).

[9] Fricke, R. and Klein, F. Vorlesungen über die Theorie der Automorphen Fuktionen, Band I. B. G. Teubner, Leipzig, 1926.

[10] Grötzsch, H. Über die Verzerrung bei schlichten nichtkonformen Abbildungen and uber eine damit zusammenhängende Erweiterung des Picardschen Satzes. Ber. Verh. sachs. Akad. Lpz., 80,503-507 (1928).

[11] Grötzsch, H. Über Verzerrung bei nichtkonformen schlichten Abbildungen mehrfach zusammenhängender Bereiche. Ber. Verh sächs. Akad. Lpz., 82,69-80 (1930).

[12] Grötzsch, H. Über möglichst konforme Abbildungen von schlichten Bereichen. Ber. Verh. sachs. Akad. Lpz., 84,114-120 (1932).

[13] Jenkins. J. A. A new criterion for quasiconformal mapping. Ann. Math. (2), 64,208-214 (1957).

[14] Kodaira, K. and Spencer. D. C. (To appear.)

[15] Lavrent'ev, M. A. Sur une classe de représentation continues. Mat. Sbornik. 42,407-423 (1935).

[16] Mori, A. On quasi-conformality and pseudo-analyticity. Trans. Amer. Math. Soc. 84,56-77 (1957).

[17] Morrey, C. B. On the solutions of quasi-linear elliptic partial differential equations. Trans. Amer. Math. Soc. 43,126-166 (1938).

[18] Pesin, I. N. Metric properties of quasi-conformal mappings. Mat. Sbornik. 40 (82), 281-294 ((1956).

[19] Pfluger, A. Quasikonforme Abbildungen and logarithmische Kapazität. Ann. Inst. Fourier, 2,69-80 (1950).

[20] Pfluger, A. (To appear.)

[21] Rauch, H. E. On the transcendental moduli of algebraic Riemann surfaces. Proc. Nat. Acad. Sci., U.S.A. 41,42-49(1955).

[22] Siegel, C. L. Uber einige Ungleichungen bei Bewegungsgruppen in der nichteuklidischen Ebene. Math. Ann. 133,127-138 (1957).

[23] Teichmüller, O. Untersunchungen über konforme and quasi-konforme Abbildungen. Dtsch. Math. 3,621-678 (1938).

[24] Teichmüller, O. Extremale quadratische quasikonforme Abbildungen and quadratische Differentiale. Preuss. Akad. Ber. 22 (1940).

[25] Teichmüller, O. Bestimmung der extremalen quasikonformen Abbildungen bei geschlossenen Riemannschen Flachen. Preuss. Akad. Ber. 4 (1943).

[26] Teichmüller, O. Beweis der analytischen Abhängigkeit des konformen Moduls einer analytischen Ringflachenschar von den Parametern. Dtsch. Math. 7,309-336 (1944).

[27] Teichmüller, O. Ein Verschiebungssatz der quasi-konformen Abbildung. Dtsch. Math. 7,336-343 (1944).

[28] Teichmüller, O. Veränderliche Riemannsche Flächen. Dtsch. Math. 7,344-359 (1944).

[29] Volkoviskii, L. I. Quasiconformal Mappings. Lwow, 1954. (Russian.)

[30] Weil, André. Sur les modules des surfaces de Riemann. Séminaire Bourbaki, May, 1958.

[31] Yujobo, Zuiman. On absolutely continuous functions of two or more variables in the Tonelli sense and quasiconformal mappings in the A. Mori sense. Comment. Math. Univ. St. Paul, 4, 67-92 (1955).

Date Due

MAY 15 '52			

DATE DUE	BORROWER'S NAME	ROOM NUMBER
MAY 15 '62	R. Bazmets	